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Partial Fourier Transform Methods to Solve the Solution Formula of Stokes Equation in Half-Space

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ABSTRAK

Fluida adalah suatu bentuk materi yang memiliki zat cair, gas, dan plasma. Dalam kehidupan sehari-hari, cairan menjadi bagian penting, seperti bagian dari darah dan juga membantu tubuh mendapatkan nutrisi. Selain itu, beberapa fenomena lingkungan terkait erat dengan mekanika fluida. Konsep fluida membantu kita memahami perilaku fluida dengan berbagai kondisi. Telah diketahui bahwa gerak fluida dapat digambarkan dalam model matematika khususnya dalam bentuk persamaan diferensial parsial (PDE) dan disebut sebagai persamaan navier stokes (NSE). Persamaan navier stokes diturunkan dari keseimbangan kekekalan massa dan kekekalan momentum. Dalam penelitian ini mempertimbangkan rumus solusi linierisasi persamaan navier stokes (NSE) dengan masalah nilai batas awal (IBV) dalam ruang setengah tanpa tegangan permukaan. Masalah model yang dipertimbangkan meliputi jenis fluida nonlinier. Prosedur penelitian yang merupakan transformasi model masalah menggunakan transformasi fourier dari sistem persamaan yang baru. Kemudian dihitung rumus solusi dari sistem persamaan baru untuk kecepatan dan kepadatan dari masalah model dengan menggunakan metode transformasi Fourier dan transformasi fourier parsial. Strategi untuk mendapatkan solusi masalah model didasarkan pada analisis beberapa penyelesaian masalah model yang diperoleh dengan menggunakan transformasi laplace dari persamaan stokes. Oleh karena itu, secara khusus, rumus kecepatan v=(v_1,…,v_N) dan kepadatan (x,t) dari persamaan stokes diperoleh.

ABSTRACT

Fluids are a shape of a matter which have substance liquids, gases and plasmas. In our daily life, fluids become important part, such as part of our blood and also help our body getting nutrients. Beside that we know that some environtmental phenomenas are intimately linked to fluid mechanics. The concept of fluid helps us understand the behavior fluid with various conditions. It is well known that fluid motion can be described in mathematical model in especially in form of partial differential equations (PDE) and called as Navier Stokes Equations (NSE). The Navier Stokes equation is derived from balance of conservation of mass and conservation of momentum. In this research consider the solution formula of the linearized of the Navier Stokes Equation (NSE) with the initial boundary value (IBV) problem in half space without surface tension. The model problem under consideration covers of non-linear fluid type. The procedures of the reseach that are transformation the model problem using Fourier transform than we have new equation system. Then we calculate the solution formula of this new equation system for velocity and density of the model problem by using Fourier transform and partial Fourier transform method. The strategy geting the solution of the model problem is based on the analysis of some resolvent of the model problem which obtained by using Laplace transform of the Stokes equations. Therefore, In particular, the formula of velocity v=(v_1,…,v_N) and density ρ(x,t) of the Stokes equation are obtained.

1. Introduction

Nowadays, studying fluid dynamic is an interesting point of view. We can investigate the characteristics of fluids motion (Huanran et al., 2022; Murena et al., 2022). Some fluids as we know are complex fluids because the substance undergo extrusion, molding and blowing processes (Sittipol et al., 2021; Zare et al., 2022). As information, one of the most common physical phenomena in nature and technological devices is fluid flow. For examples weather, circulation of blood, end so on. The study of deformation and flow of material known as rheology. There are many authors which studying not only the mathematical of the constitutive models which can analyse the characteristics of the rheology but also in numerical approaches with several methods. Fluids also apply in some applications. For example intravenous therapy in medicine.

The important thing to solve partial differential equations is the mathematical term which called well posedness (Kumar & Singh, 2021; Seadawy et al., 2019). The mathematical models of physical

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phenomena should have three properties that are solution of the model problem exists and unique, and the solution's behaviour change continuously with the initial conditions (Gawin et al., 2011; Porfyrakis & Tsitsas, 2019). Starting with the pioneering works of mathematical model of fluid motion, the Navier Stokes equation model has been studied by many authors not only in numerical methods but also in mathematical methods. In the following are researchers who concern in mathematical approaches.

Recently, many researchers studied this fields in mathematical approaches. We briefly recall several recent publications for Navier Stokes Equation (NSE). We begin with publication by (Alif et al., 2021). In the early 2021, investigated the solution formula for compressible fluid motion in three dimensional case. It is important to address the solution formula of the Stokes equation in N-dimensional case, in this paper we consider not only this case but also the solution formula of the Stokes equations in half-space case which has not been studied (Alif et al., 2021). One year before, other research studied half space model problem for compressible fluid of Korteweg type with slip boundary condition (Inna et al., 2020). Moreover, proved the local well-posedness for model fluid flow of the Oldroyd-B in bounded and unbounded domain. Many researchers interested in the local in time unique existence theorem for compressible viscous fluids in the general domain with slip boundary conditions (Inna et al., 2020; Maryani, 2016b).. They proved the maximal Lp-Lq regularity based on the boundedness of the solution operator families of the model problem.

The existence of $\mathcal{R}$-bounded solution operator families for two-phase Stokes resolvent equations have been investigated by (Maryani & Saito, 2017). The mathematical model of incompressible viscoelastic fluid of the Oldroyd type which described the steady flows has been investigated by (Baranovskii, 2019). In 2020, the research proved the solution formula and R-boundedness for the generalized Stokes resolvent problem in an infinite layer with Neumann Boundary condition (Oishi, 2021). The Lp-Lq estimates for the solutions to the linearized equations of motion of compressible viscous fluid in an exterior domain in three dimensional case and the decay estimates for the solution of the non-linear problem have to be interesting topic in fluids. In addition, many researcher concern in the fluid dynamics in numerical approaches, for example Numerical solution of the incompressible Navier Stokes equations in primitive variables. The numerical scheme which used is the lowest equal-order pair of finite element. The validity of the Navier Stokes Equations (NSE) for nanoscales liquid flows using numerical methods is investigated through by (C. Liu & Li, 2011). They consider the NSE model nanoscale through molecular dynamics simulations. The regularity class has been studied by (Saito, 2020). He proved the maximal Lp-Lq regularity for a compressible fluid model of Korteweg type in general domain. The decay properties of the Stokes equations has been investigated by (Saito & Shibata, 2016). They considered the Stokes equation with surface tension added to the model problem. Another model problem for Stokes equation in bounded domain with boundary condition and without surface tension has been investigated by (Maryani et al., 2021). For local well-posedness of free boundary problem for the Navier-Stokes equations has been investigated by (Shibata, 2018). He considers the model problem in an exterior domain. Other reseachers who concerned with Navier-Lame equation is Sakhr and Chronik (Sakhr & Chronik, 2017). They investigated the Navier-Lame in cylindrical coordinate system.

The Stokes equation with initial boundary value problem studied by (Enomoto & Shibata, 2013). They investigated the boundedness of the operator families of the model problem. In 2016, Maryani studied another model of the fluid motion that is Oldroyd-B model. She investigated the global well-posedness of the oldoryd-B Model (Maryani, 2016a; Murata & Shibata, 2020). They investigated the model problem by using maximal Lp-Lq regularity class. The time linearized of the Navier Stokes are written in the following. $\rho_t + \gamma \mathrm{div}\, \mathbf{v} = f$ in Ω, $\mathbf{v_t} - \alpha\Delta\mathbf{v} - \beta\nabla \mathrm{div}\, \mathbf{v} + \gamma\nabla\rho = \mathbf{g}$ in Ω (1) and the initial conditions $\mathbf{v}\,|_{\partial\Omega} = \mathbf{0}$, $(\rho, \mathbf{v})\,|_{t=0} = (\rho_0, \mathbf{v_0})$(1a) where α and γ are positive constants and $\alpha + \beta > 0$.

The first equation of (1) is the conservation of mass and second equation of (1) known as the balance of momentum. A detailed analysis of model problem (1) can be found in the review article (Enomoto & Shibata, 2013).

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ and with equation (1) we consider the motion of a compressible viscous fluid. Here, $\mathbf{g}$ is external force. In this paper, we consider the resolvent problem of the Stokes equations in half space. Employing the similar technique to (Enomoto & Shibata, 2013), we show the solution formula of the model problem (1). First of all, we transform the equation system of (1) using Laplace transform, then the resolvent problem of the equation (1) are being described by the set of equations. $\lambda\rho \quad + \gamma\mathrm{div} \quad = f$ in $\mathbb{R}^N_+$ $\lambda\mathbf{v} \quad - \alpha\Delta\mathbf{v} - \beta\nabla\mathrm{div}\,\mathbf{v} + \gamma\nabla\rho = \mathbf{g}$ in $\mathbb{R}^N_+$ $\mathbf{v}\,|_{x_N=0} = \mathbf{0}$ (2). Where $\mathbf{v}$ denotes the velocity of the fluid and ρ its density. We define $\mathbb{R}^N_+$ $(\mathbb{N} \geq 2)$ be the half space by. $\mathbb{R}^N_+ = \{\mathbf{x} = (x_1, x_2, \dots, x_3) \in \mathbb{R}^N \mid x_N > 0\}$. (3)

The equation system of (2) is getting by applying Laplace transformation. We can write the equation system of (2) in a matrix form. $\lambda\begin{pmatrix}\rho\\ \mathbf{v}\end{pmatrix} + \begin{bmatrix}0 & \gamma\mathrm{div}\\ \gamma\nabla & -\alpha\Delta - \beta\nabla\mathrm{div}\end{bmatrix}\begin{pmatrix}\rho\\ \mathbf{v}\end{pmatrix} = \begin{pmatrix}f\\ \mathbf{g}\end{pmatrix}$. To introduce our main result, first of all, we define the notations which are used in the whole of the paper. For a scalar-valued function $u = u(x)$ and vector-valued function $\mathbf{v}(x) = \langle v_1(x), v_2(x), \dots, v_N(x)\rangle^T$, the partial derivative with respect to x_k for $k = 1, \cdots, N$, is denoted by $\partial_k = \frac{\partial}{\partial x_k}$ and the gradient with ∇. $\nabla u = (\partial_1 u, \dots, \partial_N u)^T$, $\Delta u = \sum_{k=1}^{N} \partial_k^2 u$, $\Delta \mathbf{v} = \langle \Delta v_1, \dots, \Delta v_N\rangle^T$, $\operatorname{div} \mathbf{v} = \sum_{j=1}^{N} \partial_j \mathbf{v}_j$, $\nabla v = \{ \partial_k v_\ell \mid k, \ell = 1,2,3, \dots, N \}$, $\nabla^2 \mathbf{v} = \{ \partial_k \partial_\ell v_m \mid k, \ell, m = 1, 2, \dots, N \}$.

The set of all natural number is denoted by $\mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In whole of the paper, we denote the Laplace transform and the inverse Laplace transform by $\mathcal{L}$ and $\mathcal{L}^{-1}$, respectively, where $\mathcal{L}[f](\lambda) = \int_{\mathbb{R}} e^{-\lambda t} f(t)\, dt = \mathcal{F}_t[e^{\gamma t} f(t)](\tau)$, $\mathcal{L}^{-1}[g](t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{\lambda t} g(\tau)\, d\tau = e^{\gamma t}\mathcal{F}_t[g(\tau)](t)$, with $\lambda = \gamma + i\tau \in \mathbb{C}$. We set, $W_p^{k,m}(\Omega) = \{ \mathbb{U} = (\rho, \mathbf{v}) \mid \rho \in W_p^k(\Omega), \mathbf{v} \in W_p^m(\Omega) \}$. Before we state the main result, first of all we introduce the definition of Sobolev space $W_q^m(\Omega)$. Definition 1.1 (Adams & Fournier, 2003)

Let $k \in \mathbb{N} \cup \mathbb{N} \cup \{$ and $p \in [1, \infty)$ then the Sobolev Space $W_q^m(\Omega)$ is defined by $W_q^m(\Omega) := \{\mathbf{u} \in L_q(\Omega) \mid D^\alpha \mathbf{u}. \in L_q(\Omega), \forall\, \alpha \text{ with } |\alpha| \le m\}$.

In the following, we write conservation of mass and conservation of momentum as fundamental equations in fluid dynamics. In 1789, Antoine Lavoisier described the law of conservation of mass as a fundamental principle of physics. This law states that, despite chemical or physical transformation, mass in conserved cannot be created or destroyed within an isolated system. In other words, in a chemical reaction, the mass of the products will always be equal to the mass of the reactants. Meanwhile, Einstein amended that the law of conservation of mass-energy which describes the fact that the total mass and energy in a system remain constant. This amendment incorporates the fact that mass and energy can be converted from one to another. Following is the description of conservation of mass. The rate of change of total mass in a control volume and the net flow of mass can be written in the following. $\frac{\partial}{\partial t}\int_V \rho\, dV + \oint_S \rho\mathbf{v}\cdot\mathbf{n}\, dS = 0$. (3a). By using Gauss theorem (divergence theorem) which state in the following. Theorem 1.2 Gauss Theorem (Divergence Theorem).

Let V be a simple solid region and S is the boundary surface of V with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous first order partial derivatives and $\mathbf{n}$ be a normal vector. Then, $\oint_S \mathbf{F}\cdot\mathbf{n}\, dS = \int_V \nabla\cdot\mathbf{F}\, dV$ (3b) Applying equation (3b) to (3a), we have $\frac{\partial}{\partial t}\int_V \rho\, dV + \int_V \nabla\cdot(\rho\mathbf{v})\, dV = 0$, the derivative of the first term can be moved under the integral sign, so that we have $\int_V \frac{\partial\rho}{\partial t} dV + \int_V \nabla\cdot(\rho\mathbf{v})\, dV = 0$. (3c), Rearranging equation (3c), we have $\int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right] dV = 0$, the equation must hold for any control volume, no matter what shape and size. Therefore, the integrand must be equal to zero, $\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$. (3d). By expanding the second term of (3d), we have $\nabla\cdot(\rho\mathbf{v}) = \mathbf{v}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{v}$.. Therefore, the conservation of mass can be written in the following, $\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0$, with $\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \mathbf{v}\cdot\nabla\rho$.

In physics, it known that the conservation of momentum is total of linear momentum of a system of particles not affected by external forces. It is not depend on magnitude and direction regardless of any reactions between the parts of the system. The conservation of momentum is a fundamental concept of physics similar with the conservation of energy and conservation of mass. In physics, momentum is defined as multiplied a mass of material by the velocity of material. Following is the description of conservation of momentum. The increasing the momentum in a control volume can be constructed from net influx of momentum, body force and surface forces and we can write in the following. $\frac{\partial}{\partial t}\int_V \rho\mathbf{v}\, dV = -\oint_S \rho\mathbf{v}\,\mathbf{v}\cdot\mathbf{n}\, dS - \oint_S \mathbf{T}\cdot\mathbf{n}\, dS + \int \rho f dV$, where stress tensor $\mathbf{T}$ and deformation tensor D are defined $\mathbf{T} = (-p + \lambda\nabla\cdot\mathbf{v})\mathbf{I} + 2\mu D$, $D = \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^{\boldsymbol{T}})$, respectively. For incompressible case, the stress tensor $\mathbf{T}$ to be $\mathbf{T} = -p\mathbf{I} + 2\mu D$. The differential form of the momentum equation is derived in the following, $\frac{\partial}{\partial t}\int_V \rho\mathbf{v}\, dV = \int \rho f d + \oint_S \left(\mathbf{Tn} - \rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n})\right) dS$. By simplicity, we have $\frac{\partial}{\partial t}\int_V \rho\mathbf{v}\, dV = \int_V \rho f\, dV + \int_V \nabla\cdot(\mathbf{T} - \rho\mathbf{vv})\, dV$. (3e). Furthermore, for the advection part we have $\frac{\partial\rho\mathbf{v}}{\partial t} + \nabla\cdot(\rho\mathbf{vv}) = \mathbf{v}\frac{\partial\rho}{\partial t} + \rho\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\nabla\cdot(\rho\mathbf{v}) + \rho\mathbf{v}\cdot\nabla\mathbf{v} = \mathbf{v}\left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right) + \rho\frac{D\mathbf{v}}{Dt}$. (3f).

By using equation (3b), the equation (3f) can be written in the following $\frac{\partial\rho\mathbf{v}}{\partial t} + \nabla\cdot(\rho\mathbf{vv}) = \rho\frac{D\mathbf{v}}{Dt}$. (3g). Applying equation (3g) to equation (3e), we have $\rho\frac{D\mathbf{v}}{Dt} = \rho f + \nabla\cdot\mathbf{T}$. Following theorem is the main result of this paper. Theorem 1.3 Let $\rho, \mathbf{v}$ be density and velocity in N-dimensional Euclidean space, respectively, $\mathbb{R}^N, N \ge 2$ and set $\mathbf{x}' = (x_1, \dots, x_{N-1})^T$ and $\boldsymbol{\xi}' = (\xi_1, \dots, \xi_{N-1})^T$ then the problem (2) has a unique solution formula of $(\rho, \mathbf{v}) \in W_p^{1,2}(\mathbb{R}_+^N)$ with

$$v_j = \mathcal{F}^{-1}_{\xi'}\left[\left(\frac{\eta_\lambda i\xi_j(|\xi'|^2-A^2)}{\alpha(B^2-A^2)(AB-|\xi'|^2)} i\boldsymbol{\xi}'\cdot\boldsymbol{k}'(\xi',0)+k_j(0)\right)e^{-Bx_N} - \left(\frac{\eta_\lambda i\xi_j(|\xi'|^2-A^2)}{\alpha(B^2-A^2)(AB-|\xi'|^2)} i\boldsymbol{\xi}'\cdot\boldsymbol{k}'(\xi',0)\right)e^{-Ax_N}\right](\boldsymbol{x}',x_N)$$

$$\text{for } j=1,2,\dots,N-1 \text{ and } v_N = \mathcal{F}^{-1}_{\xi'}\left[\left(\frac{A}{(AB-|\xi'|^2)} i\xi'\cdot k'(\xi',0)\right)e^{-Bx_N} - \left(\frac{A}{(AB-|\xi'|^2)} i\boldsymbol{\xi}'\cdot \boldsymbol{k}'(\xi',0)\right)e^{-Ax_N}\right](\boldsymbol{x}',x_N)$$

$$B^2 = \alpha^{-1}\lambda + |\xi'|^2 \text{ and } A^2 = (\alpha+\eta_\lambda)^{-1}\lambda + |\xi'|^2.$$

The proof of this theorem uses Fourier transform methods for solving solution formula with operator families. Next we derive representation formulas of the solution of the reduce problem. Technical proof and the representation of the formula explained in the following section.

2. Method

Research methodology of this article is using literature review of the related articles, especially the article of (Enomoto & Shibata, 2013). In this article, we defined the solution of velocity different with their setting up the formula of velocity. In the following the procedures to find the formula of the model problem, first of all, we transform model problem (1) using Laplace transform, then we have the resolvent problem which is written in (2). By applying partial Fourier transform to equation system of (2) in half space case, we get the solution formula or velocity **v** and the density ρ. Therefore, the first and the most important step in proving Theorem 1.3 is to study the partial Fourier transform. The research procedures can be written in the following flowchart. The research procedure can be written in the flowchart presented in Figure 1.

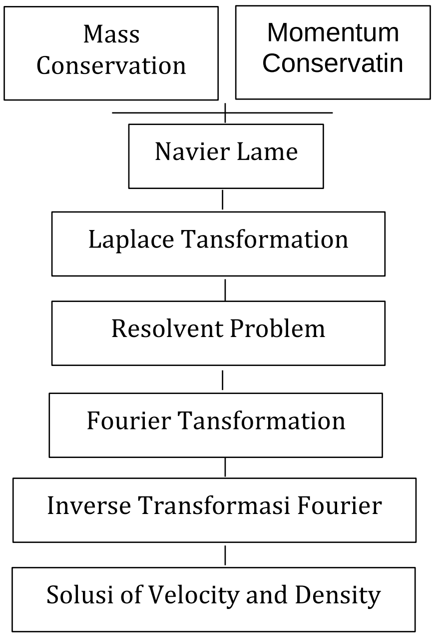

Figure 1. Flowchart of the Research

3. Result and Discussion

Results

In this section, we present a brief overview of the Stokes equation in *half space* that derived from first and second equation of (2) more closely. We prove the solution formula of system (2). Firstly, we set $\rho = \lambda^{-1}(f-\gamma \mathrm{div}\,\mathbf{v})$ in first equation of (2) and substitute ρ to the second equation of (2), we have. $\lambda\mathbf{v} - \alpha\Delta\mathbf{v} - \eta_\lambda\nabla \mathrm{div}\,\mathbf{v} = \mathbf{f}$, in $\mathbb{R}^N_+$ $\mathbf{v}\,|_{x_N=0} = 0$ (3) with $\eta_\lambda = \beta + \gamma^2\lambda^{-1}$ and $\mathbf{f} = \mathbf{g} - \lambda^{-1}\gamma\nabla f$. By using partial Fourier transform and inverse partial Fourier transform which are defined in the following,

$\boldsymbol{\mathcal{F}}_{x'}[\mathbf{u}](\xi', x_N) = \hat{\mathbf{u}}(\xi', x_N) = \int_{\mathbb{R}^{N-1}} \mathbf{u}(\mathbf{x}', x_N) e^{-i\boldsymbol{\xi}'\cdot\mathbf{x}'} d\mathbf{x}'$ and $\boldsymbol{\mathcal{F}}_{\xi'}^{-1}[\mathbf{u}](\boldsymbol{x}', x_N) = \check{\mathbf{u}}(\boldsymbol{x}', x_N) = \int_{\mathbb{R}^{N-1}} \mathbf{u}(\boldsymbol{\xi}', x_N) e^{i\boldsymbol{\xi}'\cdot\mathbf{x}'} d\boldsymbol{\xi}'$ respectively, with $\mathbf{x}' = (\mathrm{x}_1, \dots, \mathrm{x}_{N-1})^{\mathrm{T}}$ and $\boldsymbol{\xi}' = (\xi_1, \dots, \xi_{N-1})^{\mathrm{T}}$. Reducing (3) for $\mathbf{f} = \mathbf{0}$, then we have homogenous partial differential equations $\lambda\mathbf{v} - \alpha\Delta\mathbf{v} - \eta_\lambda \nabla \mathrm{div}\, \mathbf{v} = \mathbf{0}$, in $\mathbb{R}_+^{\mathrm{N}}$ (4) with initial boundary condition $\mathbf{v}\mid_{x_N=0} = k$. Applying now partial Fourier transform to resolvent problem of equation (4), for $x_N > 0$, we have $(\lambda + \alpha|\xi'|^2)v_j - \alpha D_N^2 v_j - i\xi_j \eta_\lambda (i\boldsymbol{\xi}' \cdot \mathbf{v}' + D_N v_N) = 0$ $(\lambda + \alpha|\xi'|^2)v_N - \alpha D_N^2 v_N - \eta_\lambda D_N (i\boldsymbol{\xi}' \cdot \mathbf{v}' + D_N v_N) = 0$ (5) for $j = 1, \dots, N-1$ and $i\boldsymbol{\xi}' \cdot \mathbf{v}' = \sum_{j=1}^{N-1} \xi_j v_j$. Meanwhile, for $x_N = 0$, we set $v_j \mid_{x_N=0} = k_j$, $v_N \mid_{x_N=0} = 0$, (6) for $j = 1, \dots, N-1$. Applying equation (4) by div, we have $(\lambda - (\alpha + \eta_\lambda)\Delta)\mathrm{div}\, \mathbf{v} = 0$. (7). Multiplying (4) by $(\lambda - (\alpha + \eta_\lambda)\Delta)$, then using (7), we have $(\lambda - (\alpha + \eta_\lambda)\Delta)(\lambda - \alpha\Delta)\mathbf{v} = 0$. (8). Given the Laplace operator $\Delta = \sum_{i=1}^{N-1} \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_N^2}$. (9). We applied (9) to the equation of (8), by using the partial Fourier transform and its derivation, then we have two characteristic roots for the system of ordinary differential equation (5), i.e $A = \sqrt{(\alpha + \eta_\lambda)^{-1}\lambda + |\xi'|^2}$, and $B = \sqrt{\alpha^{-1}\lambda + |\xi'|^2}$. Setting solution formula of velocity as following $v_j = (P_j + Q_j)e^{-Bx_N} - P_j e^{-Ax_N}$ (8) for $j = 1, \dots, N$. Moreover we have, $D_N^2 v_j = B^2(P_j + Q_j)e^{-Bx_N} - A^2 P_j e^{-Ax_N}$ (8a) $i\xi' \cdot \mathbf{v}' = (i\xi' \cdot P' + i\xi' \cdot Q')e^{-Bx_N} - (i\xi' \cdot P')e^{-Ax_N}$ (8b) $D_N v_N = AP_N e^{-Ax_N} - B(P_N + Q_N)\, e^{-Bx_N}$ (8c) $D_N^2 v_N = B^2(P_N + Q_N)e^{-Bx_N} - A^2 P_N e^{-Ax_N}$ (8d). $D_N(i\boldsymbol{\xi}' \cdot \mathbf{v}' + D_N v_N) = -B[(i\xi' \cdot P' + i\xi' \cdot Q') - B(P_N + Q_N)]e^{-Bx_N} + A[i\xi' \cdot P' - AP_N]e^{-Ax_N}$. (8e). Substituting equation system of (8a)-(8e) to equation system of (5) and equating the coefficients of e^{-Ax_N} and e^{-Bx_N}, then we get $i\xi' \cdot P' + i\xi' \cdot Q' = B(P_N + Q_N)$ (9), $P_j = \frac{\eta_\lambda i\xi_j (i\xi' \cdot P' - AP_N)}{\alpha(B^2 - A^2)}$. (10) By equation (9) and (10), we have $P_N = \frac{A}{AB - |\xi'|^2}(i\xi' \cdot Q' - BQ_N)$ (11), and $i\xi' \cdot P' = \frac{|\xi'|^2}{AB - |\xi'|^2}(i\xi' \cdot Q' - BQ_N)$, where $i\xi' \cdot P' = \sum_{j=1}^{N-1} i\xi_j P_j$, $i\xi' \cdot Q' = \sum_{j=1}^{N-1} i\xi_j Q_j$. Similar technique as above, by substituting equation of (8a)-(8e) to equation (6), we have the equations $Q_j = k_j(0)$ and $Q_N = k_N(0)$. (12). Multiplying first equation of (12) by $i\xi_j$ then summing up from $j = 1, \dots, N-1$, then we have $i\xi' \cdot Q' = \sum_{j=1}^{N-1} i\xi_j\, k_j(\xi', 0) = i\xi' \cdot k'(\xi', 0)$. (13). By second part of equation (12) and (13), we have the formula of $P_j = \frac{\eta_\lambda i\xi_j (|\xi'|^2 - A^2)}{\alpha(B^2 - A^2)(AB - |\xi'|^2)} i\xi' \cdot k'(\xi', 0)$ (14), $P_N = \frac{A}{(AB - |\xi'|^2)} i\xi' \cdot k'(\xi', 0)$. (15). Then, substituting equation (14) and (15) to system equation of (8), we have the solution formula of Stokes equation of (2) where $j = 1, 2, \dots, N-1$ i.e.

$$v_j = \mathcal{F}_{\xi'}^{-1}\left[\left(\frac{\eta_\lambda i\xi_j(|\xi'|^2 - A^2)}{\alpha(B^2 - A^2)(AB - |\xi'|^2)} i\xi' \cdot k'(\xi', 0) + k_j(0)\right) e^{-Bx_N} - \left(\frac{\eta_\lambda i\xi_j(|\xi'|^2 - A^2)}{\alpha(B^2 - A^2)(AB - |\xi'|^2)} i\xi' \cdot k'(\xi', 0)\right) e^{-Ax_N}\right](\boldsymbol{x}', x_N)$$

$$v_N = \mathcal{F}_{\xi'}^{-1}\left[\left(\frac{A}{(AB - |\xi'|^2)} i\xi' \cdot k'(\xi', 0)\right) e^{-Bx_N} - \left(\frac{A}{(AB - |\xi'|^2)} i\xi' \cdot k'(\xi', 0)\right) e^{-Ax_N}\right](\boldsymbol{x}', x_N),$$

which completes the proof of Theorem 1.3. Therefore, our result on solution formula of model problem (2) is obtained. For the further research we can estimate the solution operator families. This result can be a fundamental theory for the regularity of the model problem. We know that this regularity become important part for solution of the partial differential equations (PDE's). Since from regularity we can find not only local well-posedness but also global well-posedness for the model.

Discussion

In fluid dynamics, there are two interesting area to study using numerical methods. One of the most interesting area of research is to study the behaviour of turbulent for incompressible motion (Le & Ooi, 2021; Teimouri et al., 2022). Turbulent flows often occur in engineering applications, for example in the air flow around a vehicle, the water flow in turbines of hydroelectric power plants as well as in water networks and oil pipelines with high pressure. Another interesting area is studying multiphase flow, this phase occur because of two or more different fluids. There is a wide range of applications that involve multiple fluids and where numerical simulation is extensively used. In 2018, investigated the compressible-incompressible two-phase flows phase transition: Model problem (Watanabe, 2018). He studied this two-phase flows separated by a sharp interface not only with a phase transition but also with a surface tension.

Multiphase flows are a phenomenon which particularly relevant also in subsurface flow. In this area, it is often assumed that the individual fluids are well separated, that is, the fluids do not mix and there are no additional particles resolved in the fluids (J. Liu et al., 2022; Zhang et al., 2022). Other applications of multiple fluids in contact with solid are found in industrial processes. For example sintering, lab-on-a-chip processes, inkjet printing, and so on. In mathematical analysis approaches, the two phase problems of compressible and incompressible viscous fluids flow without surface tension is studied by (Kubo & Shibata, 2021). For elliptic problems with two phase incompressible flows in unbounded domain has been investigated by (Saito & Zhang, 2019). The representation of the interface as it evolves in time and the evaluation of the distributional force are two main challenges in the numerical simulation of two phase flow. There are two main strategies to couple the interface evolution problem to the Navier Stokes equations discretized on a fixed grid. One steps of the numerical approach is discretization process. In this step, interpolation is needed to describe the main variables in analysis. The analysis is based on four node, squared element to be transformed into other coordinate system. Because of the features of the proposed method, the mesh is fixed and the integration can be done analytically over one element. The interpolation functions that are used here belong to family of Lagrange elements based on *n-th* order Lagrange polynomials. The interpolation functions are make possible the use of rectangular or square elements.

Some numerical approach is used to find solution to a problem involving convection. Some methods also suitable to find the solution to any hyperbolic equation such as Naver Stokes equation system. The objectives of numerical of numerical simulations is to understand how bubbles consisting of oleaginous substances are transported and resolved in the circulation of blood. These processes are often strongly driven by surface tension, a force that strives after keeping different fluids separated. A similar application is the modelling of the absorption of air in the blood circulation in human lungs. In the early 2000's the numerical solution of the Navier Stokes Equations in three dimensional case has been investigated by many researchers. In 2010, Bikri et.al investigated the dirichlet to neumann map-an application to the Stokes model problem in half-space (Bikri et al., 2010). As we known that various methods in the numerical approach. Finite element methods is the famous methods in numerical methods. In 2012, Girault et.al using finite element methods to investigated the Navier-Stokes of the model fluid motion in theory and algorithms (Girault & Raviart, 2012). Physically, the system of (1) describes the motion of compressible fluid motion. In numerical methods, the challenging poin of view is simulation of phase field models problem in high resolution requirements. Besides that, the accurate result require sophisticated numerical approximation and fine computational meshes.

4. Conclusions and Suggestions

The solution formula of velocity and density of the model problem (2) are formed by multipliers. These multipliers are known as solution operator families. As mentioned in the end of this article, it will be a matter of future research the boundedness of the solutions operator with Neumann boundary conditions. Notice that all this result as well as the proofs can be used for future research related to fluid dynamics. From a strictly mathematical point of view, the regularity of the solution of the model problem is an important point. Beside that, this result become important step to prove boundeness in bent-half space and general domain. Therefore, by using this boundedness we can automatically regularity of the model problem which is an important point.

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