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Weakly Orthogonally Additive Functionals on The Mcshane-Bochner Integral Function Spaces Defined in Euclidean Spaces $\mathcal{R}^n$

Riyadi*

Primary School Teacher Education, Faculty of Teacher Training and Education, Universitas Sebelas Maret, Surakarta, Indonesia

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ABSTRACT

Studi tentang fungsional aditif ortogonal lemah mempunyai dampak terhadap sifat-sifat struktural dari suatu ruang fungsi dan memungkinkan penyelidikan lebih lanjut mengenai penyelesaian masalah-masalah matematika yang lebih luas. Tujuan penelitian ini adalah untuk menyelidiki sifat-sifat dan aplikasi dari fungsional aditif ortogonal lemah pada ruang fungsi terintegral McShane-Bochner yang didefinisikan di dalam ruang Euclide $\mathcal{R}^n$. Metode penelitian yang digunakan adalah Research and Development (R&D). Penelitian ini diawali dengan melakukan analisis pendahuluan melalui studi pustaka, kemudian mengujikan teori, melakukan evaluasi, dan pengambilan simpulan. Setelah itu, dilakukan pengujian teori yang telah dikembangkan melalui kegiatan Focus Group Discussion. Penelitian ini berhasil mengkonstruksi suatu ruang fungsi yang merupakan koleksi dari fungsi-fungsi yang terintegral McShane-Bochner yang didefinisikan pada sel $[\bar{a}\ \bar{b}]$ di dalam ruang Euclide $\mathcal{R}^n$ yang memenuhi sifat-sifat tertentu. Berdasarkan ruang fungsi tersebut, selanjutnya dikonstruksi Teorema Representasi untuk fungsional aditif ortogonal lemah yang didefinisikan pada ruang fungsi yang baru dikonstruksi tersebut.

ABSTRACT

The study of weakly orthogonal additive functionals has an impact on the structural properties of function space and allows further investigation into the solution of broader mathematical problems. This research aimed to investigate the properties and applications of weakly orthogonal additive functionals on the space of McShane-Bochner integrable functions defined in the Euclidean space $\mathcal{R}^n$. Research and Development (R&D) method was utilized in this study. This research began by conducting a preliminary analysis through a literature study, then testing the theory, conducting an evaluation, and drawing conclusions. After that, the developed theory was tested through Focus Group Discussion activities. This research succeeded in constructing a function space which is a collection of McShane-Bochner integral functions defined in the cell $[\bar{a}\ \bar{b}]$ in the Euclidean space $\mathcal{R}^n$ which fulfilled certain properties. Based on this function space, the Representation Theorem for the weakly orthogonal additive functionals defined in the newly constructed function space was then constructed.

1. INTRODUCTION

The topic of operators in function spaces or vector spaces is very interesting to research because operator theory has many applications, both in mathematics and in other fields such as physics, chemistry, biology, computer science and engineering, or other engineering sciences. One of the important things is to provide necessary and sufficient conditions for vector-valued composition operators in Hardy spaces (Blasco, 2020). In addition, it can be used to solve optimization problems and integral problems by utilizing the Dunford integral (Solikhin et al., 2019).

It is not known exactly when this topic began to be researched, but it began to be widely researched around the 1970s. Operator theory is developing very quickly and many new theories have been produced by mathematicians. Recently, one of the interesting topics to research is orthogonal additive operators. Several researchers have developed this topic such as Sundaresan who researched orthogonality and nonlinear functionals on Banach spaces (Sundaresan, 1972) which discussed the characterization of additive orthogonality operators in Banach space, Gumenchuk who researched the expansion of orthogonal additive operators in lattice vector spaces (Gumenchuk et al., 2014), Seng who researched about orthogonal additive operators (Chew, 1985), Lee who researched orthogonal-polynomials-based integral inequality (Lee et al., 2018). Pliev and Weber who researched C-compact orthogonally additive operators (Pliev & Weber, 2021), and Feldman conducted research on factorization for orthogonal additive operators on

*Corresponding author.
E-mail addresses: riyadi_pgsd_fkip@staff.uns.ac.id

Banach Lattice (Feldman, 2019). Orthogonal additive operator topics that are widely discussed include orthogonally additive operators in lattice-normed (Getoeva & Pliev, 2015; Abasov & Pliev, 2017, 2019; Wang et al., 2019), on orthogonally additive functions with big graphs (Baron, 2017), the lateral order on Riesz spaces and orthogonally additive operators (Mykhaylyuk et al., 2021), domination problem for narrow orthogonally additive operators (Pliev, 2017), orthogonal Birkhoff preserving additive operators (Guo, 2019), and special classes orthogonal additive operators (Abasov & Pliev, 2019; Maslyuchenko & Popov, 2019; Fotiy et al., 2020; Abasov, 2020).

Research in the field of functional analysis, especially function spaces, is an interesting topic in modern mathematics. The previous research explored semi-analytic solutions for non-linear Volterra fractional integro-differential equations using the Laplace Adomian decomposition method. The key findings demonstrate that this method is an effective tool for solving these types of equations (Alrabaiah et al., 2020). Another study examined the application of integral transforms to nonlinear dynamic models with non-integer order derivatives. The research resulted in a modified Laplace-type integral transform combined with Adomian's approach to analyze nonlinear evolution equations with non-integer derivatives (Nuruddeen et al., 2022). A method for fully discrete convergence analysis of non-linear hyperbolic equations using finite element analysis is proposed. By using second-order and fourth-order non-linear hyperbolic equations as examples, the study analyzes the full discrete convergence of these equations with the finite element method, achieving super-convergence results (Zhang, 2019).

Moreover, developing concepts and theories in function spaces is an important step in understanding the properties of functions and the relationships between mathematical objects that arise in that context. One interesting topic to research is weakly orthogonal additive functionals in the McShane-Bochner function space. This functionals refers to a mapping that maintains the additive structure and orthogonality among the functions in that space. The study of weakly orthogonal additive functionals has implications for the structural properties of function spaces and allows further investigation of the application of such concepts and properties to broader mathematical problems. Weakly orthogonal additive functionals has an important role in harmonic analysis and function theory. Therefore, its properties and characteristics can contribute to the development of functional theory and its applications in a more general functional space.

The topic of weakly orthogonal additive functionals in Mcshane-Bochner integral functional spaces has never been developed. Therefore, the researchers are interested in researching this topic. This research is a development of the research results of Rosinski and Woyczynski (Rosinski & Woyczynski, 1977), especially the development of a weakly orthogonal additive functional representation which was originally defined on a Lebesgue integral function space on the interval $[a\ b]$ in a straight line, generalized to a function space that McShane-Bochner integral defined on the cell $[\bar{a}\ \ \bar{b}\,]$ in the Euclidean space $\mathcal{R}^n$.

This research aims to further investigate the properties and applications of weakly orthogonal additive functionals on McShane-Bochner integrable functional spaces defined in Euclidean space. A deeper understanding of these functional properties is expected to contribute to the development of functional analysis and its applications in various applied mathematical contexts. In addition, this research is also directed at expanding the scope of knowledge concerning the McShane-Bochner function space, which is an increasingly important research topic in the development of modern functional analysis.

Given $\mathcal{O}$ a collection of all open intervals in $\mathcal{R}^n$, then the outer measure of any set $E \subset \mathcal{R}^n$ s a non-negative expanded real number $\mu_\alpha^*(E)$ with:

$$\mu_\alpha^*(E) = \{\inf \textstyle\sum_{i=1}^{\infty} \alpha(I_i) : I_i \in \mathcal{O} \text{ for each } i \text{ such that } E \subset \bigcup_{i=1}^{\infty} I_i\}$$

provided the infimum exists. If the infimum does not exist, then the set E is said to be unmeasurable. In this case $\alpha(I_i)$ denotes volume of I_i for each i.

As in the real number system $\mathcal{R}$, the collection of all$-\ \mu_\alpha^*$ measurable sets in $\mathcal{R}^n$ forms the $\sigma-$algebra of sets, and is denoted by Σ. The pair $(\mathcal{R}^n, \Sigma)$ is called measurable space. Next, we define the measure function $\mu: \Sigma \to \bar{\mathcal{R}}$ with $\mu(E) = \ \mu_\alpha^*(E)$ for each $E \in \Sigma$. So, if E is measurable, then $\mu(E)$ denotes the size of the set E. Furthermore, $(\mathcal{R}^n, \Sigma, \mu_\alpha)$ i.e. the measurable space $(\mathcal{R}^n, \Sigma)$ which is equipped with the measure μ is called a measure space. The measurement μ is said to be σ – finite if there exists a sequence of measurable sets $\{E_i\} \subset \mathcal{R}^n$ such that $\mathcal{R}^n = \cup_{i=1}^{\infty} E_i$ dan $\mu(E_i) < \infty$ for each i.

Given $(\mathcal{R}^n, \Sigma, \mu)$ space of measures and let v be the second measure defined in the $\sigma -$ algebra Σ. If $v(A) = 0$ as long as $\mu(A) = 0$, then we say that v is continuous absolute concerning μ and written $v \ll \mu$

Theorem 1.1 (Radon-Nikodym Theorem for Signed Measures) (Cheney, 2001)**.** Given $(\mathcal{R}^n, \Sigma, \mu)$ a space of σ – finite. If v is a signed measure that is finite and absolutely continuous with respect to μ, then there exists a measurable function h such that $v(E) = \int_E h d\mu$ or every $E \in \Sigma$.

Definition 1.2 (Pfeffer, 1993) Given α is a volume in cell E. The function $f: E \to \mathcal{R}$ s said to be McShane integrable on E if there is a number A so that for every number $\mathcal{E} > 0$ there is a positive function δ on E so that for every McShane partition δ –fine $\mathcal{P} = \{(\bar{x}, I)\} = \{(\bar{x}_1, I_1), (\bar{x}_2, I_2), \cdots, (\bar{x}_n, I_n)\}$ on E holds

$|A - \mathcal{P}\sum f(\bar{x})\alpha(I)| = |A - \sum_{i=1}^{n} f(\bar{x}_1)\alpha(I_i)| < \varepsilon$

Henceforth, the collection of all McShane integral functions in cell E is symbolized by $\mathcal{M}(E)$.

Given $E = [\bar{a}\ \bar{b}] \subset \mathcal{R}^n$ is a cell and $\mathcal{R}$ denotes the real number system. The function $s: E \to \mathcal{R}$ is said to be a simple function if there are $c_1, c_2 \cdots, c_n \in \mathcal{R}$ and $A_1, A_2 \cdots, A_n$ subsets of E that are measurable with $A_i \cap A_j = \emptyset$ for $i \neq j$ so that: $s = \sum_{i=1}^{n} c_i \chi A_i$, where χA_i is the characteristic function on A_i for each i.

Theorem 1.3 (Indrati, 2002) Given that $E = [\bar{a}\ \bar{b}] \subset \mathcal{R}^n$. If $f \in \mathcal{M}(E)$, then there exists a sequence of simple functions $\{s_n\}$ on E so that $s_n(\bar{x}) \to f(\bar{x})$ almost everywhere on E for $n \to \infty$ and it holds $lim_{n\to\infty} \int_E s_n d\alpha = \int_E f d\alpha$.

2. METHODS

This research is categorized into Research and Development (R&D) development research. Borg and Gall (2003) outlined the definition of research and development as follows:

"*R&D is an industry-based development model in which the findings of research are used to design new products and procedures, which then are systematically field-tested, evaluated, and refined until they meet specified criteria of effectiveness, quality, or similar standards".*

This research aimed to develop a new theory. This research began by conducting a preliminary analysis through a literature study, then testing the theory, carrying out evaluations, and drawing conclusions (Prakisya et al., 2022). Literature studies are carried out by reviewing concepts and the structural properties of certain concepts through books, journals, papers, and bulletins related to research problems (Agung Saputro et al., 2018; Luthfiani et al., 2019). In general, this research was carried out by studying in-depth basic theories to the most recent theories. Observations were focused on the structure of a concept, theorems, and examples. Then the theory development was carried out and tested using a Focus Group Discussion. The FGD functioned to generate new ideas to construct new, more general, and basic theories as well as receive comments and input from several experts (O.Nyumba et al., 2018). The Focus Group Discussion was conducted by mathematics experts in Indonesia, namely mathematics lecturers from various universities in Indonesia, and was attended by 30 experts. Testing the truth of these new theories was done through deductive proof.

3. RESULTS AND DISCUSSION

RESULTS

Given $E = [\bar{a}\ \bar{b}] \subset \mathcal{R}^n$ a cell and B a Banach space with norm $\|\cdot\|_B$. The function $f: E \to B$ is called a simple function if there are vectors $c_1, c_2 \cdots, c_n \in B$ and $A_1, A_2 \cdots, A_n$, namely sets of measurable subsets of E whose pairs are mutually exclusive with $\cup_{i=1}^{n} A_i = E$ so

$$f = \sum_{i=1}^{n} c_i \chi A_i$$

with $\chi A_i(\bar{x}) = 1$ if $\bar{x} \in A_i$ and 0 for others.

The function $f: E \to B$ is called strongly measurable if there exists a sequence of simple measurable functions $\{s_n\}$ such that

$$\| s_n(\bar{x}) - f(\bar{x}) \| \to 0 \ as\ n \to \infty$$

for almost all $\bar{x} \in E$

Definition 3.1 Given $E = [\bar{a}\ \bar{b}] \subset \mathcal{R}^n$ a cell, and $(B, \|\cdot\|_B)$ a Banach space.

The function $f: E \to B$ is said to be McShane-Bochner integrable on E if there exists a vector $A \in B$ such that for every $\mathcal{E} > 0$ there is a positive function δ on E such that for every δ-fine McShane partition $\mathcal{P} = \{(I, \bar{\xi})\}$ on E, it applies

$$\| \mathcal{P}\sum f(\bar{\xi})\,\alpha(I) - A \| < \varepsilon$$

Definition 3.2 Given $E = [\bar{a}\ \bar{b}] \subset \mathcal{R}^n$ a cell, $(B, \|\cdot\|_B)$ a Banach space and B^* dual Banach space of B. The function $f: E \to B$ is said to be McShane-Bochner-Pettis integrable on E if for every $I \subset E$ there is a vector $A_i \in B$ so that for every $\varepsilon > 0$ there is a positive function δ on E so that for every McShane partition McShane δ –fine $\mathcal{P} = \{(I, \bar{\xi})\}$ on E, it applies

$$| \mathcal{P}\sum x^*\left(f(\bar{\xi})\right)\alpha(I) - x^*(A_I)) | < \mathcal{E}$$

for every $x^* \in B^*$.

Henceforth, if it is not stated then what is meant by $\|\cdot\|$ is $\|\cdot\|_B$.

Definition 3.3 Given $E = [\bar{a}\ \bar{b}] \subset \mathcal{R}^n$ a cell, and $(B, \|\cdot\|_B)$ a Banach space. The function $f: E \to B$ is said to be absolutely McShane-Bochner integrable on E if f and $\|f(\cdot)\|_B$ are respectively McShane-Bochner integrable and McShane integrable on E.

Theorem 3.4 (Guoju & Schwabik, 2005) Given $E = [\bar{a}\ \bar{b}] \subset \mathcal{R}^n$ a cell, $(B, \|\cdot\|_B)$ a Banach space with norm $\|\cdot\|_B$, and $f: E \to B$ *a* function. If f is a McShane-Bochner integrable on E, then $\|f(\cdot)\|_B$ is a McShane integrable on E.

Theorem 3.5 Given $E = [\bar{a}\ \bar{b}] \subset \mathcal{R}^n$ a cell, $(B, \|\cdot\|_B)$ a Banach space, and $f: E \to B$ a function. If f is a McShane-Bochner integrable on E, then $|x^*(f(\cdot))|$ McShane-Bochner-Pettis integrable on E

Theorem 3.6 (Barra, 1981) Given $E = [\bar{a}\ \bar{b}] \subset \mathcal{R}^n$ a cell, B a Banach space, and $f: E \to B$ a function. If f is a McShane integrable on E, then f is measurably strong on E.

In the following discussion, $\mathcal{MBP}\ (E)$ represents the collection of all McShane-Bochner-Pettis integrable functions in the cell $E \subset \mathcal{R}^n$

Theorem 3.7 Given $E = [\bar{a}\ \bar{b}] \subset \mathcal{R}^n$, and $f_n, f: E \to B$ for every n.
If:
(i) $f_n \to f$ almost everywhere for $n \to \infty$ and $f_n \in \mathcal{MBP}(E)$ for every n,
(ii) $\| f_n \| \leq M$ almost everywhere on E for every n and a number $M \geq 0$,

then $f \in \mathcal{MBP}\ (E)$ and

$$\lim_{n\to\infty} \left| \int_E x^*(f_n)\, d\alpha - \int_E x^*(f)\, d\alpha \right| = 0$$

Given $\bar{b} = (b_1, b_2 \cdots, b_n) \in \mathcal{R}^n$, next, we write $\bar{b} \to \infty$, if $min_{1\leq i\leq n} b_i \to \infty$. Furthermore, if $E = [\bar{a}\ \bar{b}] \subset \mathcal{R}^n$ with $\bar{b} > \bar{1}$, then $\alpha(E) \to \infty$ provided $\bar{b} \to \infty$.

Given that $E = [\bar{1}, \bar{b}] \subset \mathcal{R}^n$ a cell, $(B, \|\cdot\|_B)$ a Banach space and B^* dual Banach of B. Furthermore, it is also known that $f \in \mathcal{MBP}\ [\bar{1}, \bar{b}]$ for every $\bar{b} > \bar{1}$, i.e. $(\mathcal{MBP}) \int_{\bar{1}}^{\bar{b}} x^*(f) d\alpha$ exists for every $\bar{b} > \bar{1}$. If $lim_{\bar{b}\to\infty}\ (\mathcal{MBP}) \int_{\bar{1}}^{\bar{b}} x^*(f) d\alpha$ exists, we define:

$$(\mathcal{MBP}) \int_{\bar{1}}^{\bar{\infty}} x^*(f)\, d\alpha = \lim_{\bar{b}\to\infty} (\mathcal{MBP}) \int_{\bar{1}}^{\bar{b}} x^*\ (f)$$

Next, we write $f \in \mathcal{MBP}[\![\bar{1}, \infty)$ if $f \in \mathcal{MBP}[\bar{1}\ , \bar{b}]$ for every $\bar{b} > \bar{1}$ and $lim_{\bar{b}\to\infty}\ (\mathcal{MBP}) \int_{\bar{1}}^{\bar{b}} x^{*(f)}\, d\alpha$ exists.

Definition 3.8 Given $E = [\bar{1}\ , \bar{b}] \subset \mathcal{R}^n$ a cell $(B, \|\cdot\|_B)$ a Banach space, B^* dual Banach of B and $f: E \to \mathcal{R}$ is a McShane-Bochner Pettis integrable function on cell E. We define the section collection $W_0(E)$ in $\mathcal{MBP}[\bar{1}, \infty)$ with:

$$W_0(E) = \{ f \in \mathcal{MBP}\ (E) : \lim_{\alpha(E)\to\infty} \frac{1}{\alpha(E)} \int_E |x^* f|\, d\alpha = 0\}$$

Theorem 3.9 $W_0(E)$ is a Banach space with respect to the norm:

$$\| f \| = \sup \{\frac{1}{\alpha(E)} \int_E |x^*(f)|\, d\alpha : E = [\bar{1}\ , \bar{b}] \text{ dengan } \bar{b} > \bar{1}\}$$

for each $f \in W_0(E)$

Proof. It is easy to show that $W_0(E)$ is a linear space and $\|\cdot\|$ is the norm on $W_0(E)$. Therefore, we will only prove that $W_0(E)$ is a complete space. Given $\{ f_n\} \subset W_0(E)$ any Cauchy sequence, namely for every number $\varepsilon > 0$ there is a natural number N_1 so that if $n, m \geq N_1$ it applies:

$\| f_n - f_m \| < \frac{\varepsilon}{\alpha(E)}$

$$\Leftrightarrow \sup\{ \{\frac{1}{\alpha(E)}\int_E |x^*(f_n - f_m)|\, d\alpha : E = [\bar{1}, \bar{b}] \text{ with } \bar{b} > \bar{1}\} < \frac{\varepsilon}{\alpha(E)}$$

$$\Leftrightarrow \frac{1}{\alpha(E)}\int_E |x^*(f_n - f_m)|\, d\alpha < \frac{\varepsilon}{\alpha(E)} \Leftrightarrow \int_E |x^*(f_n - f_m)|\, d\alpha < \varepsilon$$

This means that for every natural number m and n, there exists a number $M_{mn} \geq 0$ such that $|x^*(f_m - f_n)\ (\bar{x})| \leq M_{mn}$ is almost everywhere in E. Therefore, we can choose a natural number N_2 so that $m, n \geq N_2$ then $M_{mn} < \varepsilon$ Consequently, if $m, n \geq N_2$, it applies:

$|x^*(f_m - f_n)\ (\bar{x})| < \varepsilon$

for every $\varepsilon > 0$. This means that the sequence $\{x^* f_n(\bar{x})\}$ is a Cauchy sequence in $\mathcal{R}$. Therefore, there is a McShaneBochner-Pettis integrable function f on $E = [\bar{1}, \bar{b}]$ with $\bar{b} > \bar{1}$ so that $x^*(f_n) \to x^*(f)$ almost everywhere in E, that is, for every number $\varepsilon > 0$ there are natural numbers so that if N_3 then if $n \geq N_3$ it applies $|x^*(f_n - f)| < \varepsilon$. Based on this inequality, we have:

$$\int_E |x^*(f_n - f)| d\alpha < \int_E \varepsilon\, d\alpha = \varepsilon\, \alpha\,(E)$$

$$\Leftrightarrow \frac{1}{\alpha(E)}\int_E |x^*(f_n - f_m)|\, d\alpha < \varepsilon$$

$$\Leftrightarrow \sup\{\frac{1}{\alpha(E)}\int_E |x^*(f_n - f)|\, d\alpha\} < \varepsilon$$

$$\Leftrightarrow \|x^*(f_n - f)\| < \varepsilon$$

In other words, the sequence $\{x^*(f_n)\}$ is norm convergent to the function $x^*(f)$. Furthermore, taking $N_0 = \max\{N_1, N_2, N_3\}$ then, if $n \geq N_0$ it holds:

$$\lim_{\alpha(E)\to\infty} \frac{1}{\alpha(E)}\int_E |x^* f|\, d\alpha \leq \lim_{\alpha(E)\to\infty} \frac{1}{\alpha(E)}\int_E |x^*(f_n - f)| d\alpha + \lim_{\alpha(E)\to\infty} \frac{1}{\alpha(E)}\int_E |x^*(f_n)|\, d\alpha < \varepsilon$$

for every number $\varepsilon > 0$, this means that $f \in W_0(E)$. So $W_0(E)$ is a complete space. Therefore, $W_0(E)$ is a Banach space ■

Given $E = [\bar{1}, \bar{b}] \subset \mathcal{R}^n$ with $\bar{b} > \bar{1}$ and $\mathcal{MB}(E)$ denotes the collection of McShane-Bochner integrable functions in cell E. The sequence $\{f_n\} \subset \mathcal{MB}(E)$ is said to be bounded convergent to a function $f \in \mathcal{MB}(E)$, if $\{f_n\}$ converges point by point to the function f almost everywhere in E and (f_n) is bounded convergent almost everywhere in E. A functional $\mathcal{F}$ defined on $\mathcal{MB}(E)$ is said to be bounded continuous if $\mathcal{F}(f_n) \to \mathcal{F}(f)$ for $n \to \infty$ provided $\{f_n\}$ converges boundedly to the function f.

Given $E = [\bar{1}, \bar{b}] \subset \mathcal{R}^n$ with $\bar{b} > \bar{1}$ and B a Banach space. The function $k(\cdot,\cdot): E \times B \to B$ is said to be a Caratheodory function if $k(\bar{x}, \cdot)$ is continuous for almost all $\bar{x} \in E$ and $k(\cdot, t)$ is measurable for every t $\in B$.

Lemma 3.10 Given that $E = [\bar{1}, \bar{b}] \subset \mathcal{R}^n$ with $\bar{b} > \bar{1}$, B a Banach space, B^* dual Banach of B, and $\mathcal{MB}(E)$ denotes a collection of functions the McShane-Bochner integrable on E and $\mathcal{F}$ functional defined on $\mathcal{MB}(E)$. If $\mathcal{F}$ is boundedly continuous on $\mathcal{MB}(E)$ then $\mathcal{F}(f\, \chi_E) \to 0$ as long as $\mu(E) \to 0$ for every $f \in \mathcal{MB}(E)$

Lemma 3.11 Given $E = [\bar{1}, \bar{b}] \subset \mathcal{R}^n$ dengan $\bar{b} > \bar{1}$, B a Banach space, B^* dual Banach of B. If $\mathcal{F}$ is boundedly continuous and orthogonally additive functional on $W_0(E)$, then there is a function $k(\bar{x}, t)$: $E \times B \to B$ so that $k(\bar{x}, t)$ is McShane-Bochner integrable with respect to $\bar{x}$ in E for every $t \in B$ with $k(\bar{x}, 0) = \theta$ for almost all $\bar{x} \in E$ and holds:

$\mathcal{F}(s) = \int_E x^*(k(\cdot, s(\cdot)))d\alpha$

for all simple functions in E

Proof. Since $\mathcal{F}$ is boundedly continuous, then for every number $\varepsilon > 0$ and $t \in B$. Given $E = [\bar{1}, \bar{b}] \subset \mathcal{R}^n$ with $\bar{b} > \bar{1}$, B a Banach space, B^*dual Banach of B, $\delta(\varepsilon, t) > 0$ so that if $A \subset E$ with $\mu(A) < \delta(\varepsilon, t)$, then $\mathcal{F}(t\, \chi_A) < \varepsilon$. In other words, $\mathcal{F}$ as a set function, is absolutely continuous with respect to μ. Furthermore, if $E = \bigcup_{i=1}^{\infty} E_i$ with $E_i^o \cap E_j^o = \phi$ for i$\neq j$, then it holds:

$$\mathcal{F}(t\, \chi_E) = \lim_{n\to\infty} \mathcal{F}(t\chi \cup_{i=1}^{n} E_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathcal{F}(t\chi E_i) = \sum_{i=1}^{\infty} \mathcal{F}(t\chi E_i)$$

This means that $\mathcal{F}$ is an additive and countable set function. According to the Radon-Nikodym Theorem, there is a function $k_t^*(\cdot)$ on E so that

$\mathcal{F}(t\ x_E) = \int_E k_t^*(\cdot)d\alpha$

for each E.

Next, we define a function $k(\overline{x}, t) = k_t^*(\overline{x})$ for each $\overline{x} \in E$ and $t \in \mathrm{R}$. If $t = 0$ so $\mathcal{F}(t\ x_E) = \mathcal{F}(\theta) = 0$. Therefore, it holds $\int_E k_0^* = 0$ for every E. So $k(\overline{x}, 0) = \theta$ for almost all $\overline{x} \in E$ Furthermore, take s any simple function on E i.e. $s(\overline{x}) = \sum_{i=1}^{n} t_i \chi E_i(\overline{x})$ where E_i is a measurable set for all i and mutually exclusive pairs, and $E = \bigcup_{i=1}^{\infty} E_i$, then we get:

$$\mathcal{F}(s) = \mathcal{F}\sum_{i=1}^{n} t_i \chi E_i = \sum_{i=1}^{n} \int_{E_i} k(\cdot, t_i \chi E_i) d\alpha$$

$$\mathcal{F}(s) = \sum_{i=1}^{n} \int_{E_i} k(\cdot, s(\cdot)) d\alpha = \int_E k(\cdot, s(\cdot)) d\alpha$$

Lemma is proven ■

Lemma 3.12 Given $E = [\overline{1}\ , \overline{b}] \subset \mathcal{R}^n$ with $\overline{b} > \overline{1}$, B a Banach space, B^* dual Banach of B, $\mathcal{F}$ boundedly continuous dan weakly orthogonally additive functional on $W_0(E)$, and $k(\overline{x}, t)$: $E \times B \to B$ function obtained in Lemma 3.11. If for every number $\delta > 0$ and every boundedly closed interval $P = [-a, a]$ with $a > 0$, we define the numbers:

$W(\delta; P; E) = sup\left\{\int_E |k(\cdot, t_1) - k(\cdot, t_2)| d\alpha : t_1, t_2 \in P \text{ dan } |t_1 - t_2| < \delta\right\}$

and

$$W(\delta; P; E) = \sup\left\{\sum_{i=1}^{n} W(\delta; P; E) : \bigcup_{i=1}^{n} E_i = E, E_i^o \cap E_j^o = \phi \quad i \neq j\right\}$$

then for every interval P, $lim_{\delta \to 0^+} W(\delta; P) = 0$ applies

Lemma 3. 13 Given $E = [\overline{1}\ , \overline{b}] \subset \mathcal{R}^n$ with $\overline{b} > \overline{1}$, B Banach space, B^* dual Banach of B. If $\mathcal{F}$ is a boundedly continuous and weakly orthogonally additive functional on $W_0(E)$, then the function $k(\bar{x}, t): E \times B \to B$ i.e. the function obtained in Lemma 3.11 is a uniformly continuous function on every boundedly closed interval $P \subset \mathcal{R}$ and for every $\overline{x} \in E$

Theorem 3.14 Given $E = [\overline{1}\ , \overline{b}] \subset \mathcal{R}^n$ with $\overline{b} > \overline{1}$, $(B, \|\cdot\|_B)$ a Banach space, B^* dual Banach of B. A functional F is boundedly continuous and weakly orthogonally additive on $W_0(E)$ if and only if there exists a Caratheodory function $k(\overline{x}, t)$: $E \times B \to B$ with $k(\overline{x}, \theta) = \theta$ for all $\overline{x} \in E$ so that $k(\overline{x}, t)$ is McShane-Bochner-Pettis integrable with respect to $\overline{x}$ on E for every $t \in B$ and it holds $F(f) = \int_E x^*\left(k\left(\overline{x}, f(\overline{x})\right)\right) d\alpha$ for every function $f \in W_0(E)$ and every $x^* \in B^*$

Proof. Sufficient condition If $f \in W_0(E)$, then according to Theorem 2.28, there exists a sequence of simple functions $\{s_n\}$ on E, so that $s_n(\overline{x}) \to f(\bar{x})$ almost everywhere on E for $n \to \infty$ and without losing generality, we can assume that $|s_n(\bar{x})| \leq |f(\bar{x})|$ for all n. According to Lemma 3.13, the function $k(\overline{x}, \cdot)$ is uniformly continuous on every boundedly closed interval $P \subset \mathcal{R}$ and for every $\overline{x} \in E$. Therefore: $k(\overline{x}, s_n(\overline{x})) \to k(\overline{x}, f(\overline{x}))$ almost everywhere on E and there exists a number $M \geq 0$ such that it holds $|k(\overline{x}, f)| \leq M$ almost everywhere on E. Consequently, according to the Dominated Convergence Theorem (Theorem 2.29) the function $\lim_{n\to\infty} k(\overline{x}, s_n(\overline{x}))$ McShane-Bochner Pettis integrable on E and we get:

$$\int_E x^*(k(\cdot, f(.)) d\alpha = \lim_{n\to\infty} \int_E x^*\left(k\left(\cdot, s_n(\cdot)\right)\right) d\alpha$$
$$= \lim_{n\to\infty} \int_E \mathcal{F}(s_n)$$
$$= \mathcal{F}\left(\lim_{n\to\infty} s_n\right)$$
$$= \mathcal{F}(x^*(f))$$

Necessary conditions: Given that $f, g \in W_0(E)$ so that $f \perp g$, that it holds $f(\overline{x})g(\overline{x}) = 0$ almost everywhere E. Furthermore, we define the sets:

:
$A = \{\overline{x} \in E: g(\overline{x}) = 0\}, B = \{\overline{x} \in E: f(\overline{x}) = 0\}$ and
$C = \{\overline{x} \in E: f(\overline{x}) \neq 0\}$ dan $g(\overline{x}) \neq 0\}$
then we get:

$$\mathcal{F}(f+g) = \int_E k(\cdot,(f+g)(\cdot)d\alpha$$
$$= \int_A k(\cdot,f(\cdot))d\alpha + \int_B k(\cdot,f(\cdot))d\alpha + \int_C k(\cdot,f(\cdot))d\alpha$$
$$+ \int_A k(\cdot,g(\cdot))d\alpha + \int_B k(\cdot,g(\cdot))d\alpha + \int_C k\big(\cdot,g(\cdot)\big)d\alpha \cdot$$
$$= \int_E k(\cdot,f(\cdot))d\alpha + \int_E k(\cdot,g(\cdot))d\alpha$$
$$= \mathcal{F}(f) + \mathcal{F}(g)$$

So $\mathcal{F}$ is an orthogonally additive functional on $W_0(E)$.

Furthermore, given that $f \in W_0(E)$ and any sequence $\{f_n\} \subset W_0(E)$ such that $\{f_n\}$ converges boundedly to the function f almost everywhere in E , i.e. there exists a number $M \geq 0$ so that $|f_n(\overline{x})| \leq M$ for every n and $f_n(\overline{x}) \to f(\overline{x})$ almost everywhere on E. Since the function $k(\overline{x},\cdot)$ is continuous uniformly on every boundedly closed interval $P \subset \mathcal{R}$, and every $\overline{x} \in E$, then there exists a number $N \geq 0$ so that $\left|k\big(\overline{x}, f_n(\overline{x})\big)\right| \leq N$ for all n and almost everywhere on E. Therefore, according to the Dominated Convergence Theorem (Theorem 2.29), we obtain:

$$\lim_{n\to\infty}\int_E x^*(k\big(\cdot,f_n(\cdot)\big)d\alpha = \int_E x^*(k\big(\cdot,f(\cdot)\big)d\alpha \Leftrightarrow \lim_{n\to\infty}\mathcal{F}(f_n) = \mathcal{F}(f)$$

In other words, the functional $\mathcal{F}$ is boundedly continuous on $W_0(E)$. ■

Discussion

The results of this study demonstrated the importance of a deep understanding of the functional properties of weakly orthogonal additives. The results of this research contributed to the development of weakly orthogonal additive functional theory in the McShane-Bochner integral function space in the Euclidean space $\mathcal{R}^n$, especially in developing certain conditions that guarantee the validity of the related theorems. The properties of weakly orthogonal additive functionals proven in this study provide an important tool for mathematicians and scientists in various disciplines to analyze and model systems involving weakly orthogonal additive functionals.

The results of this research are in line with previous research on weakly additive, order-preserving, normed and positively homogeneous functionals on a metric compact (Bekzhanova, 2018). The research focused on the metrization of a space of weakly additive functionals. It began by constructing a space of weakly additive, order-preserving, normed, and positively homogeneous functionals on a metric compact. The study involved a modified Kantorovich-Rubinshtein metric within this space of weakly additive normed functionals on a metric compact. Additionally, the results of this research align with studies on the dimension of the space of weakly additive functionals (Jiemuratov, 2023). The study established weakly additive, order-preserving, normalized functionals and provides various interpretations of these functionals. Based on these findings, an example is constructed demonstrating that the space of weakly additive, order-preserving, normalized functionals cannot be embedded in any space of finite (or even countable) algebraic dimension, provided that the compact space contains more than one point.

The results of this research are not only theoretically important but also have practical implications in various fields that utilize weakly orthogonal additive functionals. An example of a possible application is in Data Analysis and Statistics. In statistics, the concept of orthogonality is used to decorrelate variables, so weakly orthogonal additive functionals can help in identifying and extracting components that are less bound or weakly correlated. Another application example is in the field of Information Theory. In coding and information theory, orthogonality is often used to maximize channel capacity and reduce interference between signals. Weakly orthogonal additive functionals can be applied to design efficient and tamper-resistant code.

It is hoped that further in-depth research can expand the application of this theorem into a wider variety of contexts. Further discussion can examine the application of these theorems in various contexts, such as applications in probability theory, signal processing, data analysis, and others. In addition, the research results provide potential for further development, including generalization to non-Euclidean spaces with more complex conditions.

4. CONCLUSION

This research has succeeded in constructing a function space which is a collection of McShane-Bochner integral functions defined in the cell $[\bar{a}\ \bar{b}]$ in the Euclidean space $\mathcal{R}^n$ which fulfills certain properties. Based on this function space, the Representation Theorem for weak orthogonal additive functionals is then constructed in the newly constructed function space.

The discussion of the Representation Theorem in this research is still limited to the Mcshane-Bochner integral function space which is defined in the Euclidean space $\mathcal{R}^n$. The discussion can be expanded in other, wider function spaces, for example in the Henstock-Bochner integral function space defined in the Euclidean space $\mathcal{R}^n$.

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